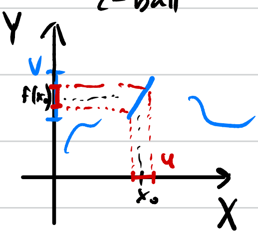
Metric Spaces and Topology
Lecture 8

Continuity. Let (X,d_X) and (Y,d_Y) be metric spaces.
A function $f: X \to Y$ is said to be **continuous at a point** $x_0 \in X$ if

$\forall$ $\underbrace{\text{neighb. } V}_{\varepsilon\text{-ball}}$ of $f(x_0)$ $\exists$ $\underbrace{\text{neighb. } U}_{\delta\text{-ball}}$ of x_0 s.t. $f(U) \subseteq V$.

$\forall x,\ d_X(x,x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$.



We say that f is **continuous** if it's continuous at every point of X.

Recall that for $f: X \to Y$ and $A \subseteq X$ and $B \subseteq Y$,

f-preimage: $f^{-1}(B) := \{x \in X : f(x) \in B\}$

f-image: $f(A) := \{y \in Y : \exists x \in A\ f(x) = y\}$.

Continuity via preimages. Let $f: X \to Y$ be a function, X, Y as above.

(a) f is continuous at $x_0 \in X$ $\Leftrightarrow$ the f-preimage of every neighb. of $f(x_0)$ is a (not necessarily open) neighb. of x_0, i.e. $\forall$ neighb. V of $f(x_0)$, $f^{-1}(V)$ is a neighb. of x_0.

(b) f is continuous $\Leftrightarrow$ f-preimages of open sets are open.

Proof. (a) f is cont. at x_0 $\iff$ $\forall$ nbh V of $f(x_0)$ $\exists$ nbh U of x_0 s.t. $f(U) \subseteq V$ $\iff$ - - - - $U \subseteq f^{-1}(V)$
$\iff$ $\forall$ nbh V of $f(x_0)$, $f^{-1}(V)$ is a nbh of x_0.

(b) $\Rightarrow$. Suppose f is cont. and let $V \subseteq Y$ be open. Let $x \in f^{-1}(V)$. Because f is cont. at x and V is a nbh of $f(x)$, $f^{-1}(V)$ must be a nbh of x. Thus, some open nbh of x is contained in $f^{-1}(V)$. Since x was arbitrary, $f^{-1}(V)$ is open.

$\Leftarrow$. Fix an arbitrary $x_0 \in X$ and let V be a nbh of $f(x_0)$, which we may assume is open by replacing V with $\operatorname{int}(V)$. Then $f^{-1}(V)$ is open by the hypothesis so it is a neighb of x_0. Hence f is cont. at x_0. $\square$

Warning. In (a) above, even if V is open, $f^{-1}(V)$ may not be because f may not be continuous at other points of $f^{-1}(V)$. For example: $f: \mathbb{R} \to \mathbb{R}$

$$x \longmapsto \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 2 & \end{cases}$$

Then $V := (1,3)$,
$f^{-1}(V) = \{0\} \cup (\frac{1}{3}, 1)$, not open.

We call $f: X \to Y$ a homeomorphism if it's a bijection and both f and f^{-1} are continuous.

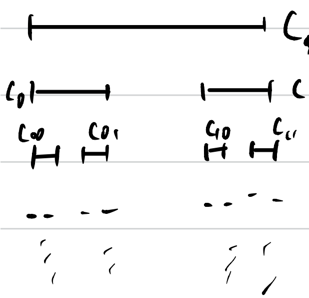
Example that the cont. of f^{-1} is not automatic.

Let $(\mathbb{R}, \tilde{d})$ be the discrete metric space and let $(\mathbb{R}, d)$ be the usual metric space and let f be the identity function $x \mapsto x$. Then f is continuous from $(\mathbb{R}, \tilde{d})$ to $(\mathbb{R}, d)$ but f^{-1} isn't.

Examples.

- $f: 2^{\mathbb{N}} \twoheadrightarrow [0,1]$ This is surjective and continuous
  $(x_n) \mapsto \underbrace{0.x_0 x_1 x_2 \ldots}_{\text{binary representation}}$ but not injective because
  $\boxed{\text{HW}}$ some rational numbers have two representations: $0.1000\ldots = 0.01111\ldots$.

- $f: 2^{\mathbb{N}} \to \mathcal{C} \subseteq [0,1]$ the Cantor set.
  $(x_n) =: x \mapsto$ the unique element in $\bigcap_n C_{x|n}$
  $$\begin{gathered} \| \\ \underbrace{0.\tilde{x}_0 \tilde{x}_1 \tilde{x}_2 \ldots}_{\text{ternary representation}}, \text{ where } \tilde{x}_n := 2 \cdot x_n. \end{gathered}$$



$\boxed{\text{HW}}$

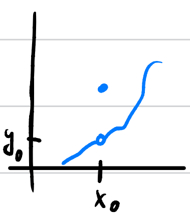
f is a homeomorphism, i.e. $2^{\mathbb{N}}$ and $\mathcal{C}$ are the same topologically (as far as open sets are concerned.

Continuity via limits. Let $f:(X,d_X)\to(Y,d_Y)$ be a function.

Obs. If $x_0\in X$ is isolated, then every function $f:X\to Y$ is continuous at x_0.

To understand continuity at nonisolated points let's look at limits.

Def. Let $f:(X,d_X)\to(Y,d_Y)$ and let $x_0\in X$. We call $y_0\in Y$ a **limit** of f as $x\to x_0$ if $\forall$ $\underbrace{\text{nbh } V}_{\varepsilon\text{-ball}}$ of y_0 $\exists$ a $\underbrace{\text{nbh } U}_{\exists\,\delta\text{-ball}}$ of x_0 s.t. $f(U\setminus x_0)\subseteq V$.



$$\Updownarrow$$

$f^{-1}(V)\cup\{x_0\}$ is a nbh of x_0.

$$\Updownarrow$$

if $0<d_X(x,x_0)<\delta$ then $d_Y(f(x),y_0)<\varepsilon$.

Obs. If x_0 is an isolated point, then $\forall y\in Y$, y is a limit of f as $x\to x_0$.

So limits only make sense for nonisolated points, in which case they are unique (in metric spaces) and we denote it by $\lim\limits_{x\to x_0} f(x)$. $\boxed{\text{HW}}$ Show uniqueness.

Limit via sequences. Let $f:(X,d_X)\to(Y,d_Y)$ and $x_0\in X$, $y_0\in Y$.
Suppose x_0 is not isolated.

$$\lim_{x\to x_0} f(x) = y_0 \iff \forall (x_n)\in X\setminus\{x_0\} \text{ if } x_n\to x_0 \text{ then } f(x_n)\to y_0.$$

Proof. $\Rightarrow$ Let $(x_n)\subseteq X\setminus\{x_0\}$ and $x_n\to x_0$. We need to show $f(x_n)\to y_0$. Fix a nbh V of y_0. We know that $U:=f^{-1}(V)\cup\{x_0\}$ is a nbh of x_0. Thus, $\forall^\infty n$ $x_n\in U$. Thus, $\forall^\infty n$ $f(x_n)\in V$.

$\Leftarrow$. We prove the contrapositive: suppose $\lim\limits_{x\to x_0} f(x)\neq y_0$. Then $\exists$ nbh V of y_0 s.t. $U:=f^{-1}(V)\cup\{x_0\}$ is not a neighbhd of x_0. Then $\forall\delta>0$, $B_\delta(x_0)\not\subseteq U$, so $\forall n$ $\exists\, x_n\in B_{\frac{1}{n}}(x_0)$ s.t. $x_n\notin f^{-1}(V)$. Then $x_n\to x_0$ but $f(x_n)\notin V$, thus, $f(x_n)\not\to y_0$. □

Continuity via limits. Let $f:(X,d_X)\to(Y,d_Y)$ and let $x_0\in X$ be a nonisolated point. TFAE:

(1) f is continuous at x_0.

(2) $\lim_{x \to x_0} f(x) = f(x_0)$.

(3) $\forall (x_n) \subseteq X$ if $x_n \to x_0$ then $f(x_n) \to f(x_0)$.

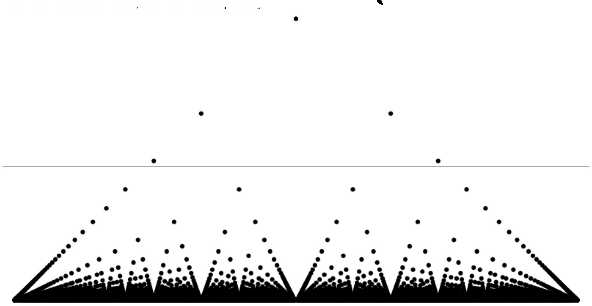
Proof. One just unravels the definitions and use the previous proposition. $\boxed{HW}$ □

Examples of continuity and discontinuity.

- Thomae's function

$$f : (0,1) \to [0, \tfrac{1}{2}]$$

$$x \mapsto \begin{cases} \frac{1}{m} & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{n}{m} \text{ reduced}. \\ 0 & \text{otherwise} \end{cases}$$



Since there are only finitely many reduced fractions $\frac{n}{m}$ in $(0,1)$ s.t. $\frac{1}{m} > \frac{1}{M}$ (for a fixed M), we see that if (x_n) is a seq. of rationals converging to an irrational, then $f(x_n) \to 0$. Thus f is cont. at irrationals. On the other hand f is discontinuous at every rational q because $\exists$ seq. $(x_n) \subseteq \mathbb{R} \setminus \mathbb{Q}$ converging to q.

This is a function which is cont. on $\mathbb{R}\setminus\mathbb{Q}$ and disc. on $\mathbb{Q}$.
Can there be an opposite function, i.e. cont. on $\mathbb{Q}$ and disc on $\mathbb{R}\setminus\mathbb{Q}$?

HW The set of continuity points of any function $f:(X,d_X)\to(Y,d_Y)$ is G_δ (= cbl intersection of opens).

Remark. We will show using Baire category or perfect set property that $\mathbb{Q}$ is not G_δ (it's F_σ by definition).

Thus, there cannot be a function $f:\mathbb{R}\to\mathbb{R}$ that is cont. on $\mathbb{Q}$ but disc at every irrational.